1\. (a) IF S IS A SET, THE POWER SET OF S IS DEFINED BY $\mathcal{P}(S) = \{A : A \subset S\}$, I.E., THE COLLECTION OF ALL SUBSETS OF S.

(b) $A \in \mathcal{P}(S)$ SIMPLY MEANS $A \subset S$.

ALL THE POWER SET DOES IS COLLECT ALL OF THE SUBSETS INTO A COLLECTION! SUBSETS OF S ARE THE SAME AS ALWAYS, BUT THEY'RE ELEMENTS OF $\mathcal{P}(S)$.

(c) IF S IS FINITE, THEN $|\mathcal{P}(S)| = 2^{|S|}$.

E.G., IF $S = \{a, b, c, d\}$, THEN $|S| = 4$ AND $|\mathcal{P}(S)| = 2^4 = 16$.
— WE CAN COUNT THE SUBSETS VIA CHOOSING WHETHER EACH ELEMENT OF S IS INCLUDED OR NOT: TWO CHOICES FOR EACH OF $|S|$ ELEMENTS GIVES $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{|S| \text{ TIMES}} = 2^{|S|}$.

2\. WHEN STUDYING A SYSTEM, THE SAMPLE SPACE IS THE SET OF POSSIBLE INDIVIDUAL OUTCOMES.

E.G., ROLLING A FAIR SIX-SIDED DIE, THE SAMPLE SPACE IS
$\Omega = \{1, 2, 3, 4, 5, 6\}$

THE EVENT SPACE IS THE SET OF ALL COMBINATIONS OF OUTCOMES, I.E., ALL SUBSETS OF THE SAMPLE SPACE.

THE EVENT SPACE IS
$\mathcal{P}(\Omega) = \{\emptyset, \{1\}, \ldots, \{1,2,3,4,5,6\}\}$

IN OTHER WORDS, AN EVENT IS SIMPLY A SET OF POSSIBLE OUTCOMES (OFTEN EXPRESSED AS A CONDITION).

"ROLL A SIX" $\rightsquigarrow \{6\}$
"ROLL AN EVEN #" $\rightsquigarrow \{2, 4, 6\}$
"ROLL A TEN" $\rightsquigarrow \emptyset$

3\. IF Ω IS A FINITE SAMPLE SPACE, WITH $|\Omega| = n \geq 1$:

(a) A PROBABILITY DISTRIBUTION ON Ω IS SIMPLY A FUNCTION THAT ASSIGNS A PROBABILITY TO EACH SAMPLE $x \in \Omega$, I.E., A FUNCTION $\mathbb{P} : \Omega \to \mathbb{R}$; IT MUST SATISFY 2 KEY AXIOMS:

① $\forall x \in \Omega$, $\mathbb{P}(x) \geq 0$, AND (PROBABILITIES MUST BE NONNEGATIVE)

② $\sum_{x \in \Omega} \mathbb{P}(x) = 1$. (EXACTLY ONE OUTCOME OCCURS)

THIS NOTATION SIMPLY SAYS TO ADD UP (Σ IS A CAPITAL SIGMA, READ AS "SUM") ALL OF THE VALUES $\mathbb{P}(x)$ FOR EACH $x \in \Omega$.

WE THEN ASSIGN PROBABILITIES TO EVENTS $A \subset \Omega$ BY ADDING UP THE PROBABILITIES OF THE SAMPLES THAT EVENT CONTAINS:

$$\mathbb{P}(A) = \sum_{x \in A} \mathbb{P}(x)$$

(WHICH ALLOWS US TO RESTATE ② AS $\mathbb{P}(\Omega) = 1$)

* NOTE: IN FULL-BLOWN PROBABILITY THEORY, THE SAMPLE SPACE Ω NEED NOT BE A FINITE SET, AND NOT EVERY SUBSET OF Ω MIGHT BE PERMITTED AS AN EVENT; IN THIS CASE, PROBABILITIES ARE ONLY DEFINED FOR EVENTS (NOT SAMPLES), AND ONE MORE AXIOM IS NECESSARY. WE WILL NOT GO THERE IN CSCI 200 — IF YOU'RE INTERESTED, MATH 310 PRESENTS A BROADER VIEW!

(b) IN A UNIFORM PROBABILITY DISTRIBUTION ON Ω, ALL SAMPLES ARE EQUALLY LIKELY, I.E. $\forall x, y \in \Omega$, $\mathbb{P}(x) = \mathbb{P}(y)$

(SO ② TELLS US THAT FOR ANY SAMPLE $y \in \Omega$, $1 = \sum_{x \in \Omega} \mathbb{P}(x) = \sum_{x \in \Omega} \mathbb{P}(y) = |\Omega| \cdot \mathbb{P}(y)$,

AND THUS $\mathbb{P}(y) = \frac{1}{|\Omega|}$).

(i) FOR ANY SAMPLE $x \in \Omega$, $\mathbb{P}(x) = \frac{1}{|\Omega|}$, AS ABOVE

(ii) FOR ANY EVENT $A \subset \Omega$, $\mathbb{P}(A) = \sum_{x \in A} \mathbb{P}(x) = \sum_{x \in A} \frac{1}{|\Omega|} = |A| \cdot \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$.

4. PROBLEM 3(b)(ii) GIVES US AN APPROACH TO ANY BASIC PROBABILITY QUESTION IN THE CASE OF A UNIFORM PROBABILITY DISTRIBUTION (ON A FINITE SAMPLE SPACE Ω):

IF $A \subset \Omega$, TO COMPUTE $\mathbb{P}(A)$, JUST COUNT THE ELEMENTS OF A,
COUNT THE ELEMENTS OF Ω,
AND DIVIDE!

5. ROLLING TWO FAIR SIDED DICE (IN ORDER) HAS $6 \times 6 = 36$ POSSIBLE OUTCOMES, EACH OF WHICH IS EQUALLY LIKELY (THUS EACH HAS PROBABILITY $\frac{1}{36}$):

FIRST DIE

SECOND DIE	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

BLOCKS ARE SAMPLES — THE SUMS OF THE DICE ARE IN RED

(a) BOTH DICE SHOW THE SAME #: $\frac{6}{36}$

(b) THE FIRST DIE SHOWS A # STRICTLY GREATER THAN THE SECOND: $\frac{15}{36}$

(c) PROBABILITY OF EACH SUM:

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

6. FOR 10 CONSECUTIVE FLIPS OF A FAIR COIN, WE HAVE TWO CHOICES (H OR T) FOR EACH FLIP, 10 TIMES, GIVING $\underbrace{2\cdot 2\cdot \cdots \cdot 2}_{10} = 2^{10} = \underline{1024}$ SAMPLES IN OUR SAMPLE SPACE.

THE CORRESPONDING EVENT SPACE HAS $\underline{2^{1024}} \approx 1.8 \times 10^{308}$ ELEMENTS!

(a) FIRST FLIP IS H LEAVES 2^9 CHOICES FOR THE REST, GIVING $\frac{2^9}{2^{10}} = \frac{1}{2}$

(b) IDENTICAL FIRST & SECOND FLIPS GIVES HH··· OR TT···, SO 2 CHOICES FOR THE FIRST PAIR OF FLIPS, LEAVING 2^8 CHOICES FOR THE REST:

$$\frac{2\cdot 2^8}{2^{10}} = \frac{1}{2}$$

(c) DIFFERENT FIRST & SECOND FLIPS GIVES HT··· OR TH···, SO 2 CHOICES FOR THE FIRST PAIR OF FLIPS, LEAVING 2^8 CHOICES FOR THE REST:

$$\frac{2\cdot 2^8}{2^{10}} = \frac{1}{2}$$

(d) ALL FLIPS ARE T GIVES JUST ONE OUTCOME, $\overbrace{TT\cdots T}^{10}$. THUS, THE PROBABILITY IS $\frac{1}{2^{10}} = \frac{1}{1024}$.

(e) FIRST 9 FLIPS ARE T, AND THE LAST IS H ALSO GIVES JUST ONE OUTCOME, SO THE PROBABILITY IS $\frac{1}{2^{10}} = \frac{1}{1024}$.

(f) HTHTHTHTHT: JUST AS IN (d)&(e), ONLY ONE OUTCOME, SO THE PROBABILITY IS $\frac{1}{2^{10}} = \frac{1}{1024}$.

(g) H_k: EXACTLY k H'S: FOR ANY GIVEN k, COUNTING THE OUTCOMES MEANS CHOOSING WHERE TO PUT k H'S AMONG 10 ROLLS, I.E., $\binom{10}{k}$.

$\mathbb{P}(H_k)$ IS THUS $\frac{\binom{10}{k}}{2^{10}} = \frac{\binom{10}{k}}{1024}$.

NOT CANCELLING THE 2'S IN THE DENOMINATOR, THIS GIVES:

k	0	1	2	3	4	5	6	7	8	9	10
$\mathbb{P}(H_k)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

(h) 3 H'S IN THE FIRST FIVE ROLLS, THEN 2 H'S IN THE LAST FIVE ROLLS:

AS ABOVE, WE NEED TO CHOOSE 3 OF THE FIRST 5 TO BE H: $\binom{5}{3}$,
THEN CHOOSE 2 OF THE NEXT 5 TO BE H: $\binom{5}{2}$.

THE TOTAL IS $\binom{5}{3}\cdot\binom{5}{2} = 10\cdot 10 = 100$, SO THE PROBABILITY IS $\frac{100}{1024}$.

COMPARE TO $\frac{252}{1024}$ ABOVE — 3+2 IS JUST ONE WAY TO GET 5 H'S (IN FACT, ABOUT 40% OF THE WAYS).